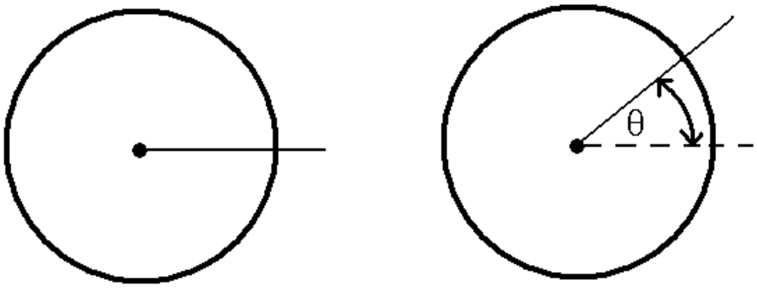
Chapter 6: Angular Motion

In this chapter we shall extend our discussion of motion to objects which travel in circular paths. This type of motion is frequently encountered in nature and a knowledge of it is necessary for an understanding of such diverse topics as the motion of bodies in the solar system and golf.

A. Angular Distance θ

Suppose the wheel mounted on an axle through its center, as shown below, is rotated through and angle θ. There are three different ways in which this rotation is measured.



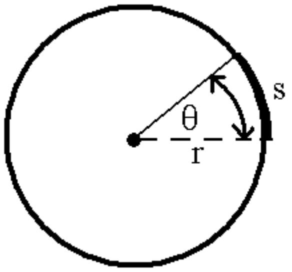
The two common ways to measure θ are with revolutions and degree units. A revolution is defined to be on complete turn, and one complete turn is defined to be 360 degrees. These two units are simply related by

$$1 \text{ rev} = 360^\circ$$

Although you are probably most familiar with these two methods for measuring angular displacement, it is important that you realize that both of these units are arbitrary. There is no physical reason why there should be 360 degrees in one revolution. We could divide a complete revolution into 10, 20, 100, or 1000 parts and define a different relationship between revolutions and degrees. The reason for using 360˚ in one revolution has historical roots.

<u>Question</u>: Can you think of a reason why ancient scientists and mathematicians chose to divide a revolution into 360 degrees?

The third way to measure θ is not arbitrary. It is called <u>radian measure</u> and is defined in terms of a ratio of two important lengths: the radius r and the arc length s. If we observe below the wheel of radius r as it is rotated by an angle θ, we see that the rim of the wheel has moved through an arc of length s.



Obviously, these three variables are related. If θ is increased so is s. Likewise, if r is increased so is s. The relationship between these three variables can be represented as a simple equation without any proportionality constant of we choose the correct unit for measuring θ. This unit is called radian measure, and we find

$$\theta = \frac{s}{r} \tag{6.1}$$

Where θ is measured in radians (rad).

Since s = 2πr for one full turn of the wheel, we see that

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad}$$

Note: s/r is simply a ratio of lengths, so, strictly speaking, it has no units. Even so, we shall say, for example, "the angle θ is π rad (radians) or 180˚ or ½ rev" to make it clear how we are measuring angles.

Examples:

1. A certain angle is 90˚. Find its equivalent in radians and revolutions.

$$\theta = 90^\circ = 90^\circ \left(\frac{2\pi\ rad}{360^\circ}\right) = \frac{\pi}{2}\, rad$$

$$\theta = 90^\circ = 90^\circ \left(\frac{1\, rev}{360^\circ}\right) = \frac{1}{4}\, rev$$

2. A certain angle is 1 radian. Find its equivalent on degrees and revolutions.

$$\theta = 1\, rad = 1\, rad \left(\frac{360^\circ}{2\pi\ rad}\right) = 57^\circ$$

$$\theta = 1\, rad = 1\, rad \left(\frac{1\, rev}{2\pi\ rad}\right) = 0.16\, rev$$

B. Angular Velocity ω

When we say that a record is rotating at 33 1/3 rev/min (rpm), we are giving its angular speed. We are telling how far it rotates in a given amount of time. The average angular velocity of a rotating object is defined to be the angular distance divided by the time taken to turn through this angle. The defining equation for average angular velocity is

$$\overline{\omega} = \frac{\theta}{t} \tag{6.2}$$

Where θ is the angle through which an object rotates in time t (ω is the Greek letter "omega"). As we see, the units for ω are those of an angle divided by a time. For example, the units might be degrees per second, revolutions per minute, or radians per second.

The definition for average velocity is very similar to our definition of average velocity for linear motion. In the case of linear motion we have $v = d/t$, where d is the linear distance moved in time t. In what follows, we shall see that each of our linear motion equations has an analog in circular motion.

When we discussed linear motion, we realized that there is an important difference between average and instantaneous velocity. This distinction also exists in angular motion. We shall always write average angular velocity as "$\overline{\omega}$" and instantaneous angular velocity as "ω".

C. Angular Acceleration α

Recall that average linear acceleration was defined as the rate at which velocity changes.

$$\overline{a} = \frac{v_F - v_I}{t}$$

Rotating objects may also experience changes in velocity. When this occurs we speak of the average angular acceleration $\overline{\alpha}$ (Greek "alpha"). This quantity we speak of is defined as the rate at which velocity changes and the defining equation is analogous to the linear acceleration equation.

$$\overline{\alpha} = \frac{\omega_F - \omega_I}{t} \tag{6.3}$$

Example: When a bowling ball is first released it slides down the alley before it starts rolling. If it takes 1.2 seconds for a bowling ball to attain an angular velocity of 6 rev/sec, determine the average angular acceleration of the bowling ball.

$$\overline{\alpha} = \frac{\omega_F - \omega_I}{t} = \frac{6\,rev/\sec - 0}{1.2 \sec} = \frac{5\,rev/\sec}{\sec} = 5\,rev/\sec^2$$

Note: The units of angular acceleration are angular velocity divided by time. The answer in the previous example would be read "five revolutions per second per second," or simply "five revolutions per second squared."

D. Angular Motion Equations with Constant Angular Acceleration

If the angular acceleration is uniform, we have that the average acceleration is the same as the instantaneous angular acceleration.

$$\overline{\alpha} = \alpha$$

And, as in the case of linear motion, the average angular velocity is given by

$$\overline{\omega} = \tfrac{1}{2}(\omega_F + \omega_I)$$

Once again there is a striking similarity between linear and rotational motion. It is now a simple matter to demonstrate that, in analogy with the equations of motion for constant linear acceleration connecting d, v, a, and t, we have exactly the same equations for rotational motion with constant angular acceleration, except that θ, ω, and α have replaced d, v, and a.

Linear	Angular
$v_F = v_I + at$	$\omega_F = \omega_I + \alpha t$
$d = \tfrac{1}{2}(v_F + v_I)t$	$\theta = \tfrac{1}{2}(\omega_F + \omega_I)t$
$d = v_I t + \tfrac{1}{2}at^2$	$\theta = \omega_I t + \tfrac{1}{2}\alpha t^2$
$v_F^2 = v_I^2 + 2ad$	$\omega_F^2 = \omega_I^2 + 2\alpha\theta$

<u>Example</u>: A skater initially turning at 3 rev/sec slows down with constant angular deceleration and stops in 4 seconds. Find her angular deceleration and the number of revolutions she makes before stopping.

This is a typical angular motion problem.

$$Know\begin{cases}\omega_I = 3rev/\sec \\ \omega_F = 0 \\ t = 4\sec\end{cases}$$

Now find α.

$$\omega_F = \omega_I + \alpha\, t$$

$$\alpha = \frac{\omega_F - \omega_I}{t} = \frac{0 - 3\; rev/\sec}{4 \sec} = -0.75\; rev/\sec^2$$

The negative sign indicates a deceleration. The number of revolutions she makes can now be determined.

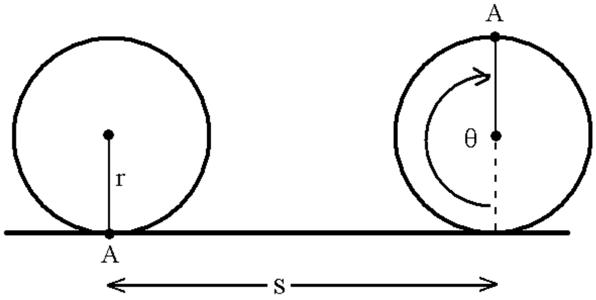
$$\theta = \omega_I t + \tfrac{1}{2}\alpha t^2$$

$$\theta = (3rev/\sec)(4\sec) + \tfrac{1}{2}(-0.75 rev/\sec^2)(4\sec)^2$$

$$\theta = 12rev - 6rev = 6rev$$

E. Connection Between Rotational and Translational Motion

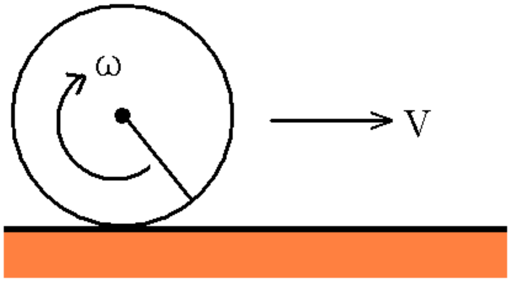
Up to this point we have treated linear and rotational motion separately. However, there are many cases, such as a ball rolling on the ground, where both rotational and linear motion occurs. To investigate this relationship, examine the wheel of radius r that has rolled a distance s as shown below.



A reference point (A) on to the rim of the wheel has been identified. We first observe that the arc length traced out by our reference point is equal to the distance s that the arc wheel moves. This distance is called the tangential distance. The relation between s and θ has already been used in our definition of radian measure.

Tangential distance = $s = r\theta$

We will now use this equation and the defining equation of angular velocity to determine a relationship between linear and rotational velocity. To do this we consider a wheel rotating with constant angular velocity ω as it moves to the right with linear velocity v.



First define speed of any point on the rim of the wheel as the tangential speed v_t. Clearly, all points on the rim of the wheel have the same tangential speed. Furthermore, it should be obvious that the tangential speed is equal to the linear velocity v of the wheel. The angular velocity of the wheel is defined by

$$\omega = \frac{\theta}{t}$$

θ is the angular displacement in time t and it is related to the distance the wheel travels by $s = r\theta$. Substituting $\theta = s/r$ into this equation we have

$$\omega = \frac{\theta}{t} = \frac{s/r}{t} = \left(\frac{s}{t}\right)\left(\frac{1}{r}\right)$$

But s/t is the linear velocity of the wheel that we have seen is also equal to the tangential velocity of any point on the rim of the wheel. So

$$\omega = v_t\left(\frac{1}{r}\right)$$

$$or,$$

$$Tangential\ speed\ = v_t\ = r\omega$$

Finally, examine the relation between linear and rotational acceleration. If we examine the previous results ($s = r\theta$ and $v_t = \omega r$) a reasonable guess would be that our derivation will yield $a_t = r\alpha$. If an object is experiencing a constant angular acceleration then both ω and v_t must be changing. From our definition of constant angular acceleration we have

$$\alpha = \frac{\omega_F - \omega_I}{t} = \frac{(v_{tF}/r) - (v_{tI}/r)}{t}$$

$$\alpha = \frac{1}{r}\left(\frac{v_{tF} - v_{tI}}{t}\right)$$

The quantity in parentheses is the rate at which tangential velocity is changing. This quantity is called the <u>tangential acceleration</u> a_t. It is the rate of change in tangential velocity of any point in the rim. In addition, it is also the linear acceleration of the wheel. The result, as predicted, is

$$\alpha = a_t\left(\frac{1}{r}\right)$$

$$or,$$

$$Tangential\ \text{acceleration} = a_t\ = r\alpha$$

<u>Notes</u>:

1. Whenever we use the equations for tangential quantities, <u>all</u> angular quantities must be in radian measure.

2. θ, ω, and α are the same for <u>all</u> points on a rotating rigid body.

3. s, v_t, and a_t are not the same for all points on a rotating rigid body. These three quantities all depend on the distance from the center of rotation.

F. Centripetal Force

In Chapter 3 Newton's first law of motion was stated as "A body continues to move at a constant velocity unless acted upon by a force." Remembering that "constant velocity" means traveling at constant speed in a straight line, we realize that Newton knew that circular motion is not natural. A force is required to maintain circular motion. When a car turns a corner this force is provided by the friction between the tires and the road. When a ball is twirled at the end of a string this force is provided by the tension on the string. And as the earth orbits around the sun this force is provide by the gravitational attraction between the two bodies.

In these three cases the force required to continue circular motion is directed toward the center of the circle. It is called centripetal force and is defined by

$$Centripetal\ Force = F_{cent} = \frac{mv_t^2}{r}$$

Since the velocity of an object changes when it moves in a circle we must have an acceleration. An acceleration requires an unbalanced force, which in this case is the centripetal force. We use Newton's second law to define this centripetal acceleration.

$$Centripetal\ Force = F_{cent} = ma_{cent} = \frac{mv_t^2}{r}$$

$$where\ Centripetal\ acceleration = a_{cent} = \frac{v_t^2}{r}$$

Note: Even when an object travels in a circle with constant tangential speed, the object experiences an unbalanced force (the centripetal force) and it is accelerating (the centripetal acceleration).

Example: A 1200 kg (2,640 lb) car is turning a corner at a speed of 8 m/sec (18 mph), and it travels along the arc of a circle in the process. If the radius of this circle is 9 m, what is the centripetal force required to hold the car in the circular path?

$$F_{cent} = \frac{mv_t^2}{r} = \frac{(1200\ kg)(8\ m/\sec)^2}{9\ meters} = 8530\ N$$

This force must be supplied by the friction force if the pavement of the tires. You can show that the coefficient of friction must be at least 0.73 in order to provide such a large force. If the pavement is wet or icy so that there is little friction between the tires and the road, the friction force on the tires will perhaps not be this large. In that event the car will skid out of a circular path (into more nearly a straight line) and may not be able to negotiate the curve.

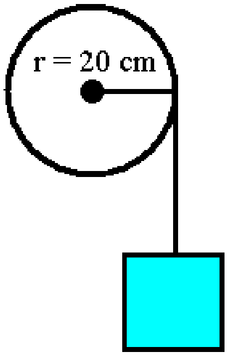
Summary

Angular distance
Arc length
Radian measure
Angular velocity
Angular acceleration
Motion with constant angular acceleration
Tangential distance
Tangential velocity
Tangential acceleration
Centripetal force
Centripetal acceleration

Problems

1. How many degrees and radians are equivalent to 3 revolutions?

2. Through what angle must the wheel shown below turn in order to unwind 40 cm of string?



3. Find the average angular velocity of a pitcher's arm if, in throwing the ball, his arm rotates one-third of a revolution on 0.1 sec.

4. If a record rotates at a constant speed of 33 1/3 rpm, how many revolutions does it make in 1 sec?

5. A turntable rotating at 33 1/3 rpm (0.56 rev/sec) is shut off. It brakes with constant angular deceleration and stops in 26 seconds. Find the angular acceleration and the number of revolutions it makes.

6. The time required for a roulette wheel to coast to rest is 15 seconds. If the wheel completed nine revolutions in this time what was the initial angular velocity?

7. A bicyclist starts from rest and accelerates at 2 rad/sec^2. In 10 seconds what is the angular velocity of the bike wheels and how many revolutions has the bike wheels completed?

8. The bike wheels in problem 7 have a radius of 0.4 m. Find the linear speed, linear acceleration, and the distance covered in 10 seconds.

9. A golfer swings a nine iron (radius = 3 ft) and a driver (radius = 4 ft) with a maximum angular velocity of 5 rad/sec. Find the tangential velocity at the clubhead.

10. Find the angular velocity of a pitcher's arm (radius = 70 cm) when he releases a 90 mph (40 m/sec) fastball.

11. A runner on a circular track is subject to the same type of centripetal force described in the discussion of the automobile. Calculate the centripetal force on a 70 kg runner with a speed of 6.7 m/sec (this is a 4 minutes mile) on a track of radius 15 m.

12. Calculate the centripetal force on a running back as he tries to make a sharp cut. Assume his mass is 80 kg and his velocity is 10 m/sec when he turns with a radius of 1 m.

13. Five students are ice stating. They hold hands while skating with a velocity of 10 sec and suddenly attempt to skate in a circle. If the outer most student has a mass of 70 kg, travels with tangential velocity of 10 m/sec and is a distance of 8 meters from the center of rotation, calculate the centripetal acceleration and force acting on this student.

14. An old trick is to hold a pail of water with your hand and swing it in a vertical circle. If the rotation rate is large enough, the water will not fall out of the pail when it is upside down at the top of its path. What is the minimum angular velocity your hand must have if this trick is to succeed? Assume the combined length of your arm and the pail is 1 meter.